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Tensor Product of Generalized Sample Spaces

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The tensor product of generalized sample spaces or manuals is defined within the framework of empirical logic. The requirement to accurately reflect the interaction of experimental procedures for coupled systems leads to three levels of product: the cross-product, operational product, and tensor product. The structure of the weights on these products is examined and is used to give a condition for the existence of the tensor product. Categorical properties of the tensor product, including a universal mapping theorem, are given.

1. INTRODUCTION

Empirical logic, as developed by Foulis, Randall, and others, is a precise language in which physical theories can be expressed, compared, and evaluated. The primary structure in this language is a generalization of the classical notion of a sample space for an experiment. This forms the basis for a generalization of conventional statistics, called *operational statistics*, and provides a formalism for dealing with subjects such as quantum mechanics where methods of classical statistics often prove to be inadequate. This research program is motivated by the work of Kolmogorov in probability and statistics (Kolmogorov, 1956), and the work of Dirac and von Neumann in quantum mechanics (Dirac, 1958; von Neumann, 1955). Comparisons between empirical logic/operational statistics and other approaches to the foundations of empirical science can be found in Randall and Foulis (1978).

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In any of these theories, it is important to have an effective method for combining two physical systems. This "product" of systems should be motivated by the physical properties of actual experiments and should, in addition, be based on a solid mathematical foundation. It soon becomes apparent that to satisfy such criteria we need not one but several different products which can reflect any influence (or lack of influence) between the systems. Thus, in Section 3, we present the cross-product, operational product, and tensor product of generalized sample spaces to allow for the presence of bilateral, unilateral, or no influence, respectively.

The definitions of products for our formalism were introduced by Foulis and Randall at the Marburg Conference on the Interpretations and Foundations of Quantum Theories in 1979. For the sake of completeness we will repeat the basic definitions and some immediate consequences here, and refer the reader to the proceedings of this conference (Foulis and Randall, 1979; Randall and Foulis, 1979) for a more detailed discussion of the motivation for these definitions.

2. BASIC DEFINITIONS

In obtaining our generalized sample space, we start with a nonempty collection of nonempty sets $\mathscr{A}$. The sets $E \in \mathscr{A}$ are called *operations* and thought of as possible results of some physical experiment. Thus elements $x \in E$ are called *outcomes* of E and the set $X = \{x\colon\ x \in E \text{ and } E \in \mathscr{A}\}$, sometimes written $\cup \mathscr{A}$, is the set of all $\mathscr{A}$ outcomes. Following the terminology of classical probability, a subset of an operation is called an *event* and the collection of all $\mathscr{A}$ events is denoted $\mathscr{E}(\mathscr{A})$. Note that an arbitrary subset $A \subseteq X$ is not considered an event unless $A \subseteq E$ for some $E \in \mathscr{A}$.

This leads to a generalization of the notion of mutually exclusive events. For A and B in $\mathscr{E}(\mathscr{A})$, we say that A is *orthogonal* to B, and write $A \perp B$, if $A \cap B = \phi$ and there is an $E \in \mathscr{A}$ with $A \dot{\cup} B \subseteq E$. (We will use the notation $\dot{\cup}$ to indicate a disjoint union.) In the special case where $A \perp B$ and $A \dot{\cup} B = E \in \mathscr{A}$, we call A and B *operational complements*, denoted $A \operatorname{oc} B$.

Notice that we have made no restrictions on operations overlapping in our structure and, in fact, it is precisely this identification of outcomes (and hence events) from different operations that provides the richness of the theory. One may think of the collection $\mathscr{A}$ as a set of classical sample spaces (operations) where an event A may appear in more than one sample space. This defines a natural "equivalence" as follows: We say two events A and C are *operationally perspective*, denoted $A \operatorname{op} C$, if there is a $B \in \mathscr{E}(\mathscr{A})$ with

$A \operatorname{oc} B$ and $C \operatorname{oc} B$. Such a common operational complement B is called an *axis of perspectivity*. The only condition imposed on our collection of sets $\mathscr{A}$ is that the op-relation be a true equivalence relation:

Definition 2.1. A nonempty collection of nonempty sets $\mathscr{A}$ is called a *generalized sample space* or *manual* if for any A, B, C in $\mathscr{E}(\mathscr{A})$ with $A \operatorname{op} C$ and $C \operatorname{oc} B$, we have $A \operatorname{oc} B$.

The terms "generalized sample space" and "manual" are used interchangeably; the first emphasizes the structure's relationship to classical probability and statistics, while the second reminds us of its role as a collection of physically realizable experimental procedures.

A *premanual* is a nonempty collection of nonempty sets which is contained in a manual. For any premanual $\mathscr{A}$, there is a unique smallest manual containing $\mathscr{A}$. This is called the *manual generated by* $\mathscr{A}$, denoted $\langle \mathscr{A} \rangle$, and is simply the intersection of all manuals which contain $\mathscr{A}$. It is important to note here that $\langle \mathscr{A} \rangle$ will always have the same outcome set as $\mathscr{A}$.

We will need the following finiteness definitions in what follows:

Definition 2.2. Let $\mathscr{A}$ be a manual (or premanual).

(i) $\mathscr{A}$ is called *finite* if it contains a finite number of operations.

(ii) $\mathscr{A}$ is called *locally finite* if each operation has a finite number of outcomes.

(iii) $\mathscr{A}$ is called *totally finite* if it is both finite and locally finite.

Three obvious examples of manuals are the *classical* manual, the *transformation* manual, and the *semiclassical* manual. A classical manual is one with a single operation; a transformation manual is one in which each operation has a single outcome, and a semiclassical manual is one in which no two operations intersect. Further examples, notably the Hilbert space manual and Borel space manual, may be found in Foulis and Randall (1979) and Randall and Foulis (1979).

3. PRODUCTS OF MANUALS

We now consider the problem of how to combine two manuals to form a product manual. Let $\mathscr{A}$ and $\mathscr{B}$ be manuals with outcome sets X and Y, respectively. We think of $\mathscr{A}$ as representing a collection of possible experimental operations for one observer (or physical system), while $\mathscr{B}$ represents a second observer with a (possibly different) set of procedures. In each of the product manuals, the outcome set will be $X \times Y$, and we will denote product outcomes with juxtaposition, xy. Similarly, if $A \in \mathscr{E}(\mathscr{A})$ and $B \in \mathscr{E}(\mathscr{B})$, $AB = \{xy\colon x \in A \text{ and } y \in B\}$.

The simplest of the products to consider is the *cross-product* $\mathscr{A} \times \mathscr{B} = \{EF\colon E \in \mathscr{A} \text{ and } F \in \mathscr{B}\}$. We think of the two observers acting with no knowledge of the other; the first executing some operation $E \in \mathscr{A}$ while the second chooses any $F \in \mathscr{B}$ to execute. If $x \in E$ and $y \in F$ results, the outcome xy would be reported for the product operation.

In an *operational product*, one of the observers is allowed to execute an operation first. The outcome of this operation determines exactly which operation the other observer performs. There are two operational products: $\overrightarrow{\mathscr{A}\mathscr{B}}$ consists of all such operations initiated with an $\mathscr{A}$ operation, and $\overleftarrow{\mathscr{A}\mathscr{B}}$ consists of all such operations initiated with a $\mathscr{B}$ operation. This yields the following definition:

Definition 3.1. An *operational product* of manuals $\mathscr{A}$ and $\mathscr{B}$ is either
(i) $\overrightarrow{\mathscr{A}\mathscr{B}} = \{\bigcup_{e \in E} eF_e\colon E \in \mathscr{A} \text{ and } F_e \in \mathscr{B} \text{ for all } e \in E\}$
(ii) $\overleftarrow{\mathscr{A}\mathscr{B}} = \{\bigcup_{f \in F} E_f f\colon F \in \mathscr{B} \text{ and } E_f \in \mathscr{A} \text{ for all } f \in F\}$

It is not hard to check that each of $\mathscr{A} \times \mathscr{B}$, $\overrightarrow{\mathscr{A}\mathscr{B}}$, $\overleftarrow{\mathscr{A}\mathscr{B}}$ is a manual, given manuals $\mathscr{A}$ and $\mathscr{B}$. Note that any operation in $\mathscr{A} \times \mathscr{B}$ can be found in both $\overrightarrow{\mathscr{A}\mathscr{B}}$ and $\overleftarrow{\mathscr{A}\mathscr{B}}$, and, in fact, $\mathscr{A} \times \mathscr{B} = \overrightarrow{\mathscr{A}\mathscr{B}} \cap \overleftarrow{\mathscr{A}\mathscr{B}}$. We wish to point out that we are taking a small liberty in writing this as an equality since an operation EF has a temporal order when considered as an element of $\overrightarrow{\mathscr{A}\mathscr{B}}$ or $\overleftarrow{\mathscr{A}\mathscr{B}}$ and this is not present in $\mathscr{A} \times \mathscr{B}$.

In constructing the tensor product $\mathscr{A} \otimes \mathscr{B}$, we require that it contain both $\overrightarrow{\mathscr{A}\mathscr{B}}$ and $\overleftarrow{\mathscr{A}\mathscr{B}}$, and, of course, that it be a manual. In fact, we define it to be the smallest such manual. For convenience, we let $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ denote $\overrightarrow{\mathscr{A}\mathscr{B}} \cup \overleftarrow{\mathscr{A}\mathscr{B}}$.

Definition 3.2. The *tensor product* of manuals $\mathscr{A}$ and $\mathscr{B}$, denoted $\mathscr{A} \otimes \mathscr{B}$, is defined to be $\langle \overleftrightarrow{\mathscr{A}\mathscr{B}} \rangle$, if it exists.

There are three possibilities for $\overleftrightarrow{\mathscr{A}\mathscr{B}}$: it may be a manual, it may be a premanual but not a manual, or it may not even be a premanual. While the vast majority of cases fall in the second category, any of the three possibilities may occur. Clearly, if $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ is a manual, it is equal to $\mathscr{A} \otimes \mathscr{B}$. This happens only for small classes of manuals $\mathscr{A}$ and $\mathscr{B}$, as described in the following theorem.

> *Theorem 3.3.* For manuals $\mathscr{A}$ and $\mathscr{B}$, $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ is a manual if and only if any one of the following three conditions hold:
> (i) $\mathscr{A}$ or $\mathscr{B}$ is a classical manual.
> (ii) $\mathscr{A}$ or $\mathscr{B}$ is a transformation manual.
> (iii) $\mathscr{A}$ and $\mathscr{B}$ are semiclassical manuals.

Proof. Assume that none of the conditions (i)–(iii) hold. Without loss of generality, we may assume that $\mathscr{A}$ is not semiclassical and $\mathscr{B}$ is neither classical nor a transformation manual. We show that $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ could not be a manual. Since $\mathscr{A}$ is not semiclassical there must be nonempty disjoint events $A, B, C \in \mathscr{E}(\mathscr{A})$ such that $A \dot{\cup} B$ and $B \dot{\cup} C$ are operations in $\mathscr{A}$. In $\mathscr{B}$ there must be events $Q, R, U, V \in \mathscr{E}(\mathscr{B})$ with $Q \dot{\cup} R$ and $U \dot{\cup} V$ operations in $\mathscr{B}$, $Q \neq U$ and only V possibly empty. Then in $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ we have the following operations:

$$AQ \cup AR \cup BU \cup BV \in \overrightarrow{\mathscr{A}\mathscr{B}}$$

$$AQ \cup BQ \cup CR \cup BR \in \overleftarrow{\mathscr{A}\mathscr{B}}$$

$$AQ \cup AR \cup BQ \cup BR \in \mathscr{A} \times \mathscr{B}$$

Therefore we have $(AQ \cup BU \cup BV)\mathrm{op}(AQ \cup BQ \cup BR)$ with axis AR, and $(AQ \cup BQ \cup BR)\mathrm{oc}\, CR$. If $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ were a manual, we would have $E = AQ \cup BU \cup BV \cup CR \in \overleftrightarrow{\mathscr{A}\mathscr{B}}$. However, $E \in \overrightarrow{\mathscr{A}\mathscr{B}}$ implies $A \cup B \cup C \in \mathscr{A}$ and $E \in \overleftarrow{\mathscr{A}\mathscr{B}}$ implies $Q \cup U \cup V \cup R \in \mathscr{B}$, neither of which is possible.

To show the converse, one need only show that each of cases (i), (ii), (iii) produces a manual $\overleftrightarrow{\mathscr{A}\mathscr{B}}$. It is not hard to show that if $\mathscr{A}$ is classical, we have $\overleftrightarrow{\mathscr{A}\mathscr{B}} = \overrightarrow{\mathscr{A}\mathscr{B}}$, and if $\mathscr{A}$ is a transformation manual, we have $\overleftrightarrow{\mathscr{A}\mathscr{B}} = \overleftarrow{\mathscr{A}\mathscr{B}}$. The proof for case (iii) is quite complicated, and is omitted here. ■

We next exhibit manuals $\mathscr{A}$ and $\mathscr{B}$ such that $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ is not a premanual. Fortunately, this occurs only in extreme examples where both of the manuals exhibit undesirable statistical and logical properties. Reasonably mild conditions on either the weights or the logics of $\mathscr{A}$ or $\mathscr{B}$ (some of which will be discussed in the next section) ensure that $\mathscr{A} \otimes \mathscr{B}$ exists for all commonly considered classes of manuals.

Example 3.4. Let $\mathscr{A}$ be the "wedge" manual: $\mathscr{A} = \{\{x, y, z\}, \{x, u, r\}, \{y, v, r\}, \{z, w, r\}\}$. Then $\mathscr{A} \otimes \mathscr{A}$ does not exist.

In cases where $\mathscr{A} \otimes \mathscr{B}$ exists, computing the smallest manual containing $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ usually proves to be an enormous task. For example, the Wright triangle [$\mathscr{A} = \{\{a, d, b\}, \{b, e, c\}, \{a, f, c\}\}$] tensored with itself has 423 operations. An APL computer routine may be employed to find the size of the tensor product in many cases. However, an important open question is to find workable methods for constructing the tensor product of two arbitrary generalized sample spaces. Some progress has been made in this area by studying the probability measures (weight functions) associated with a manual. Contrary to the complicated nature of the tensor product itself, the weight functions supported by the tensor product of generalized sample spaces are quite well behaved, easily described, and furnish compelling evidence as to the practical value of the products.

4. WEIGHT FUNCTIONS ON GENERALIZED SAMPLE SPACES AND THEIR PRODUCTS

Much of the theory of classical probability deals not with the form of particular sample spaces but with the set of probability measures which can be put on those spaces. Analogously, a central role in the analysis of a given generalized sample space $\mathscr{A}$ is given to its set of weight functions, denoted by $\Omega(\mathscr{A})$. A *weight* on a generalized sample space $\mathscr{A}$ with outcome set X is a function w: $X \to [0,1]$ such that $\sum_{x \in E} w(x) = 1$ for any operation $E \in \mathscr{A}$. Thus a weight is merely a discrete probability measure on each of the operations in $\mathscr{A}$ which agrees on any outcomes where the operations may overlap. The set of all weights on $\mathscr{A}$ may be thought of as the set of all complete stochastic models for the physical system described by $\mathscr{A}$. Any weight may be extended to a function on $\mathscr{E}(\mathscr{A})$ by defining $w(A) = \sum_{x \in A} w(x)$ for any $A \in \mathscr{E}(\mathscr{A})$. One can easily check that ω satisfies most reasonable axioms for a probability measure once they have been modified to fit into the framework of generalized sample spaces. For example, to say that w is finitely additive for a set of events $A_1, A_2, \ldots, A_n$, we must assume that the events are jointly orthogonal, i.e., pairwise orthogonal and contained in one common operation.

Let us now turn to the problem of describing the weight functions allowed by each of our products. In the following discussion we will assume that both $\mathscr{A}$ and $\mathscr{B}$ are manuals with outcome sets X and Y, respectively, and that $\mathscr{A} \otimes \mathscr{B}$ exists.

If $\alpha \in \Omega(\mathscr{A})$ and $\beta \in \Omega(\mathscr{B})$, we may define the *product weight* $\alpha\beta$: $XY \to [0,1]$ by $\alpha\beta(xy) = \alpha(x)\beta(y)$. It is easily shown that a product weight is a valid weight on any of our products and one readily sees that the set of product weights represent those stochastic models under which the operations in $\mathscr{A}$ are statistically independent of those in $\mathscr{B}$.

Note that adding operations to a manual increases the number of constraints on potential weight functions (assuming the outcome set remains the same), and hence $\mathscr{A}_1 \subseteq \mathscr{A}_2$ implies $\Omega(\mathscr{A}_2) \subseteq \Omega(\mathscr{A}_1)$. Recalling the order of inclusion from Section 3, then, we see that $\Omega(\mathscr{A} \otimes \mathscr{B}) \subseteq \Omega(\overleftrightarrow{\mathscr{A}\mathscr{B}}) \subseteq \Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$ and $\Omega(\overleftarrow{\mathscr{A}\mathscr{B}}) \subseteq \Omega(\mathscr{A} \times \mathscr{B})$. Although $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ is generally not a manual, it is still reasonable to consider the set of weight functions allowed by that collection of operations.

Fortunately, even though we may have very little knowledge of the actual operations in $\mathscr{A} \otimes \mathscr{B}$, we can still say a great deal about the weight functions on $\mathscr{A}\otimes\mathscr{B}$. This is made possible by Theorem 4.2. In order to prove this theorem, we must understand the construction of the manual generated by a premanual. This is exhibited by the following proposition, which is not hard to prove:

Proposition 4.1. Let $\mathscr{A}$ be a premanual, $X=\cup\,\mathscr{A}$. Define $\mathscr{A}^{(1)}=\{S\subseteq X$: there exist $A,B,C\in\mathscr{E}(\mathscr{A})$ with $S=A\dot{\cup}C$ and $A\operatorname{op}B$, $B\operatorname{oc}C$ in $\mathscr{A}\}$. We iterate this process by defining $\mathscr{A}^{(n)}=\{S\subseteq X$: there exist $A,B,C\in\mathscr{E}(\mathscr{A}^{(n-1)})$ with $S=A\dot{\cup}C$ and $A\operatorname{op}B$, $B\operatorname{oc}C$ in $\mathscr{A}^{(n-1)}\}$. Then $\langle\mathscr{A}\rangle=\bigcup_{n=1}^{\infty}\mathscr{A}^{(n)}$.

Theorem 4.2. If $\mathscr{A}$ is a premanual, $\Omega(\mathscr{A})=\Omega(\langle\mathscr{A}\rangle)$.

Proof. It is easily seen that $w\in\Omega(\langle\mathscr{A}\rangle)$ implies $w\in\Omega(\mathscr{A})$. Conversely, let $w\in\Omega(\mathscr{A})$. By the above construction of $\langle\mathscr{A}\rangle$, it suffices to show $w\in\Omega(\mathscr{A}^{(n)})$ for each n. We use induction on n. Assume $w\in\Omega(\mathscr{A}^{(n-1)})$. Let $S\in\mathscr{A}^{(n)}$. Then there exist $A,B,C\in\mathscr{E}(\mathscr{A}^{(n-1)})$ with $S=A\dot{\cup}C$ and $A\operatorname{op}B$, $B\operatorname{oc}C$ in $\mathscr{A}^{(n-1)}$. Therefore $w(A)=w(B)=1-w(C)$ and we see that $w(S)=w(A)+w(C)=1$. Therefore $w\in\Omega(\mathscr{A}^{(n)})$ and we are done. ∎

It is not hard to see that $\Omega(\overleftrightarrow{\mathscr{A}\mathscr{B}})=\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})\cap\Omega(\overleftarrow{\mathscr{A}\mathscr{B}})$, and hence we see that $w(\mathscr{A}\otimes\mathscr{B})=\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})\cap\Omega(\overleftarrow{\mathscr{A}\mathscr{B}})$. Therefore we will know all of the possible weights on $\mathscr{A}\otimes\mathscr{B}$ if we can find all weights on the much simpler operational products. This analysis is aided further by a very nice (and physically appropriate) characterization of the weights which are supported by an operational product.

Let w be a weight on $\mathscr{A}\times\mathscr{B}$. We seek necessary and sufficient conditions for w to be in $\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$. For any operation $E\in\mathscr{A}$ we may define ${}_Ew$: $Y\to[0,1]$ by ${}_Ew(y)=w(Ey)$. Note that ${}_Ew$ is a sort of marginal probability and it is easily checked that ${}_Ew\in\Omega(B)$. We refer to ${}_Ew$ as the weight w *preconditioned* by the operation E. Similarly, for any $F\in\mathscr{B}$, we may define the *postconditioned* weight $w_F\in\Omega(\mathscr{A})$ by $w_F(x)=w(xF)$ for all $x\in X$.

Now suppose that E and G are two different operations in $\mathscr{A}$. For an arbitrary weight $w\in\Omega(\mathscr{A}\times\mathscr{B})$, we have no *a priori* constraint that ${}_Ew={}_Gw$. In fact, it is quite possible to have ${}_Ew\neq{}_Gw$, which would imply that the execution of a particular operation in $\mathscr{A}$ has some discernable "influence" on the probabilities of $\mathscr{B}$ outcomes. It is precisely this form of influence which we find is eliminated in weights on the operational product.

Theorem 4.3. Let $\mathscr{A}$ and $\mathscr{B}$ be manuals and $w\in\Omega(\mathscr{A}\times\mathscr{B})$. Then (i) $w\in\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$ if and only if $w_F=w_H$ for all $F,H\in\mathscr{B}$, and (ii) $w\in\Omega(\overleftarrow{\mathscr{A}\mathscr{B}})$ if and only if ${}_Ew={}_Gw$ for all $E,G\in\mathscr{A}$.

Proof. Let $w\in\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$ and $F,H\in\mathscr{B}$. Let $x\in X$. We must show $w(xF)=w(xH)$. Let E be any operation in $\mathscr{A}$ containing x. Then EF and

$(E-x)F \cup xH$ are both operations in $\overrightarrow{\mathscr{A}\mathscr{B}}$. Hence $\omega(xF)=1-w((E-x)F)=w(xH)$, and we see that $w_F=w_H$. Conversely, let $w \in \Omega(\mathscr{A}\times\mathscr{B})$ and suppose $w_F=w_H$ for all $F, H \in \mathscr{B}$. Let $K=\bigcup_{e\in E} eF_e$ be an operation in $\overrightarrow{\mathscr{A}\mathscr{B}}$. We must show $w(K)=1$. Let F be any operation in $\mathscr{B}$. By assumption, $w(eF_e)=w(eF)$ for every $e \in E$. Thus $w(K)=w(\bigcup_{e\in E} eF_e)=\sum_{e\in E} w(eF_e)=\sum_{e\in E} w(eF)=w(EF)=1$, and we are done. ■

In other words, the weights allowed on $\overrightarrow{\mathscr{A}\mathscr{B}}$ are precisely those in which $\mathscr{B}$ operations have no observable influence on $\mathscr{A}$ operations, and vice versa for weights on $\overleftarrow{\mathscr{A}\mathscr{B}}$. Combining this with Theorem 4.2 we have the following obvious corollary.

Theorem 4.4. Let $\mathscr{A}$ and $\mathscr{B}$ be manuals such that $\mathscr{A}\otimes\mathscr{B}$ exists. A weight $w \in \Omega(\mathscr{A}\times\mathscr{B})$ is a weight on $\mathscr{A}\otimes\mathscr{B}$ if and only if ${}_Ew={}_Gw$ for all $E, G \in \mathscr{A}$ and $w_F=w_H$ for all $F, H \in \mathscr{B}$.

It is in this sense that we claimed at the end of Section 1 that our three levels of products would allow for the possibility of bilateral influence (cross-product), unilateral influence (operational product), or no influence (tensor product). However, one should be careful to note that we are not claiming that there can be no correlation between the two systems. In general, there are many weights on $\mathscr{A}\otimes\mathscr{B}$ which are not of the form of product weights $\alpha\beta$ and hence may describe different degrees of dependence among the systems without being influenced by a particular choice of operations.

5. STATISTICAL CONDITIONS FOR EXISTENCE

Clearly, any results pertaining to premanuals also give us insight into the tensor product. The most useful criteria which has been developed to date for determining if an arbitrary collection of sets $\mathscr{A}$ is a premanual is based on its set of weights $\Omega(\mathscr{A})$.

Theorem 5.1. Suppose $\mathscr{A}$ is a nonempty collection of nonempty sets and for any $x \in \cup\mathscr{A}$, there exists $w \in \Omega(\mathscr{A})$ with $w(x)>\frac{1}{2}$. Then $\mathscr{A}$ is a premanual.

Proof. Let $\mathscr{B}=\{E \subseteq \cup\mathscr{A}\colon \sum_{x\in E} w(x)=1 \text{ for all } w \in \Omega(\mathscr{A})\}$. Clearly, $\mathscr{A}\subseteq\mathscr{B}$. The proof will be completed by showing that $\mathscr{B}$ is a manual. Let $A, B, C \in \mathscr{E}(\mathscr{B})$ with $A \operatorname{op} B$, $B \operatorname{oc} C$ in $\mathscr{B}$. For any $w \in \Omega(\mathscr{A})$, it is easy to see that $w(A)=w(B)=1-w(C)$. Hence, $w(A)+w(C)=1$, and it remains to show $A \cap C=\varnothing$. Let $x \in A \cap C$. By assumption, there is $w \in \Omega(\mathscr{A})$ with $w(x)>\frac{1}{2}$. Therefore, $w(A)>\frac{1}{2}$ and $w(C)>\frac{1}{2}$, a contradiction. Thus $A \dot{\cup} C \in \mathscr{B}$, and we are done. ■

In general, the manual $\mathscr{B}$ produced by the procedure in the proof will not be $\langle \mathscr{A} \rangle$. Furthermore, there is an abundance of premanuals (and manuals) which fail to satisfy the conditions of Theorem 5.1.

Let us now return to the existence question for the tensor product. Recall that for $\alpha \in \Omega(\mathscr{A})$, $\mathscr{B} \in \Omega(\mathscr{B})$, the resulting product weight $\alpha\beta$ is always in $\Omega(\mathscr{A} \otimes \mathscr{B})$. Therefore, we get the following result as an immediate corollary to the above theorem.

Corollary 5.2. Let $\mathscr{A}$, $\mathscr{B}$ be manuals and suppose for each $x \in \cup \mathscr{A}$, there exists $\alpha \in \Omega(\mathscr{A})$ with $\alpha(x) > 1/\sqrt{2}$, and for each $y \in \cup \mathscr{B}$, there exists $\beta \in \Omega(\mathscr{B})$ with $\beta(y) > 1/\sqrt{2}$. Then $\mathscr{A} \otimes \mathscr{B}$ exists.

It is interesting to note that if $\mathscr{A}$ is the wedge manual of Example 3.4, the largest weight which can be put on the outcome r is $2/3$, which is just less than $1/\sqrt{2}$, so we should not expect to improve greatly on that bound. However, again note that there are many examples for which the criteria of Corollary 5.2 are not satisfied and $\mathscr{A} \otimes \mathscr{B}$ still exists.

It is not necessary for both manuals to be well behaved to ensure that the tensor product exists. There is a class of particularly nice manuals which may be tensored with any other manual. We will need the following definitions. The *dispersion-free* (d.f.) weights on a manual $\mathscr{A}$, denoted $\Omega_{\text{df}}(\mathscr{A})$, are the weights which assign only the values 0 or 1 to any outcome. The d.f. weights represent the deterministic models on a generalized sample space. A set of weights $\Delta \subseteq \Omega(\mathscr{A})$ is called *unital* if for every $x \in \cup \mathscr{A}$, there is a $w \in \Delta$ with $w(x) = 1$. Finally, $\mathscr{A}$ is called a *udf manual* if it has a unital set of dispersion-free weights. These are our "particularly nice" manuals, and we have the following theorem:

Theorem 5.3. If $\mathscr{A}$ is a udf manual, then $\mathscr{A} \otimes \mathscr{B}$ exists for any manual $\mathscr{B}$.

We will need some additional terminology: For $w \in \Omega(\mathscr{A})$, define $w^1 = \{x \in \cup \mathscr{A}: w(x) = 1\}$. Then for any manual $\mathscr{A}$, we define $\mathscr{A}^* = \{w^1: w \in \Omega_{\text{df}}(\mathscr{A})\}$. Notice that $\mathscr{A}^*$ might be empty; however, if $\mathscr{A}$ is udf, then $\cup \mathscr{A}^* = \cup \mathscr{A}$. We will discuss certain maps between manuals, called "interpretation" morphisms, in the next section. Each such morphism $\mathscr{A} \xrightarrow{\varphi} \mathscr{B}$ defines a relation on the outcome sets $X \times Y$. The set $\text{INT}(\mathscr{A} \to \mathscr{B})$ is the set of all such relations. Given this background, the proof of the above theorem rests on the following sequence of facts, each of which is itself of major importance.

Facts 5.4. (a) If $\mathscr{A}$ is udf, then $\mathscr{A}^*$ is a manual. (2) If $\mathscr{A}$ and $\mathscr{B}$ are manuals, then INT($\mathscr{A} \to \mathscr{B}$) is a manual. (3) If $\mathscr{A}$ is udf, then $\overleftrightarrow{\mathscr{A}\mathscr{B}} \subseteq$ INT($\mathscr{A}^* \to \mathscr{B}$).

6. THE TENSOR PRODUCT AS A CATEGORICAL OBJECT

In the category of manuals, the appropriate morphisms are those maps which "pull back" each weight on the range manual to a weight on the domain manual. That is, we wish to define a map $\mathscr{A} \xrightarrow{\varphi} \mathscr{B}$ such that for every $w \in \Omega(\mathscr{B})$, we have $w \circ \varphi \in \Omega(\mathscr{A})$. It is easy to check that the following definition satisfies this criterion:

Definition 6.1. Let $\mathscr{A}$ and $\mathscr{B}$ be manuals with outcome sets X and Y, respectively. A map φ: $X \to \mathscr{P}(Y)$ is called an *interpretation* if the following two conditions are satisfied: (i) (operation-preserving): For every $E \in \mathscr{A}$, $\varphi(E) \in \mathscr{B}$; (ii) ($\perp$-preserving): For every $A, B \in \mathscr{E}(\mathscr{A})$ with $A \perp B$, $\varphi(A) \perp \varphi(B)$ in $\mathscr{B}$.

We view φ as a map $\mathscr{E}(\mathscr{A}) \to \mathscr{E}(\mathscr{B})$ and write $\varphi(A)$ for $\cup_{x \in A} \varphi(x)$, as in the definition above.

Extending this notion to a cross-product of manuals, we wish to define a "bi-interpretation" to be a map $\mathscr{A} \times \mathscr{B} \xrightarrow{\varphi} \mathscr{C}$ such that every weight on $\mathscr{C}$ will induce via φ a weight on $\mathscr{A}$ and a weight on $\mathscr{B}$. Such a map will be defined on the outcome sets and lifted as before to a map on all events.

Definition 6.2. Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be manuals with outcome sets X, Y, and Z, respectively. A map φ: $X \times Y \to \mathscr{P}(Z)$ is called a *bi-interpretation* if the following two conditions are satisfied:

(i) For any $E \in \mathscr{A}$ and $\{F_e : e \in E\} \subseteq \mathscr{B}$, $\varphi(\cup_{e \in E} eF_e) \in \mathscr{C}$, and for any $F \in \mathscr{B}$ and $\{E_f : f \in F\} \subseteq \mathscr{A}$, $\varphi(\cup_{f \in f} E_f f) \in \mathscr{C}$

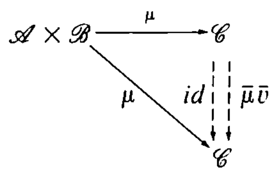
(ii) For every $A_1, A_2 \in \mathscr{E}(\mathscr{A})$ with $A_1 \perp A_2$, $\varphi(A_1 B_1) \perp \varphi(A_2 B_2)$ for any $B_1, B_2 \in \mathscr{E}(\mathscr{B})$, and for every $B_1, B_2 \in \mathscr{E}(\mathscr{B})$ with $B_1 \perp B_2$, $\varphi(A_1 B_1) \perp \varphi(A_2 B_2)$ for any $A_1, A_2 \in \mathscr{E}(\mathscr{A})$.

If $\mathscr{A} \times \mathscr{B} \xrightarrow{\varphi} \mathscr{C}$ is as bi-interpretation and $w \in \Omega(\mathscr{C})$, we define w_1 on X by $w_1(x) = w(\varphi(xF))$ for any $F \in \mathscr{B}$, and we define w_2 on Y by $w_2(y) = w(\varphi(Ey))$ for any $E \in \mathscr{A}$. The conditions on a bi-interpretation are precisely what is required to ensure that these definitions are well defined, and that $w_1 \in \Omega(\mathscr{A})$ and $w_2 \in \Omega(\mathscr{B})$.

Theorem 6.3. (universal mapping theorem). Let $\mathscr{A}$ and $\mathscr{B}$ be manuals such that $\mathscr{A} \otimes \mathscr{B}$ exists. Then there exists a unique pair $(\mu, \mathscr{C})$

satisfying (i) $\mathscr{C}$ is a manual; (ii) $\mathscr{A} \times \mathscr{B} \xrightarrow{\mu} \mathscr{C}$ is a bi-interpretation; (iii) for every manual $\mathscr{M}$ and bi-interpretation $\mathscr{A} \times \mathscr{B} \xrightarrow{\varphi} \mathscr{M}$, there is a unique interpretation $\mathscr{C} \xrightarrow{\bar{\varphi}} \mathscr{M}$ such that $\varphi = \bar{\varphi}\mu$. Furthermore, $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$ as defined earlier.

Proof. Uniqueness: Let $(\mu, \mathscr{C})$, $(\upsilon, \mathscr{D})$ be two such pairs. By hypothesis, then, there exist interpretations $\mathscr{C} \xrightarrow{\bar{\upsilon}} \mathscr{D}$ and $\mathscr{D} \xrightarrow{\bar{\mu}} \mathscr{C}$ such that $\upsilon = \bar{\upsilon}\mu$, and $\mu = \bar{\mu}\upsilon$. Then $\bar{\mu}\bar{\upsilon}(\mu) = \bar{\mu}(\bar{\upsilon}\mu) = \bar{\mu}\upsilon = \mu$ and, by uniqueness of the completion on the diagram below, $\bar{\mu}\bar{\upsilon} = id(\mathscr{C})$:



Similarly, $\bar{\upsilon}\bar{\mu} = id(\mathscr{D})$, and uniqueness is proved.

Existence: Let $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$, and define μ by $(x, y) \mapsto xy$. Then clearly (i) and (ii) are satisfied. Let $\mathscr{A} \times \mathscr{B} \xrightarrow{\varphi} \mathscr{M}$ be a bi-interpretation into a manual $\mathscr{M}$. Clearly, $\bar{\varphi}$ must be defined by $\bar{\varphi}(xy) = \varphi(x, y)$, and hence $\bar{\varphi}$ is unique. We must show that $\bar{\varphi}$ is an interpretation. It is not hard to show that $\bar{\varphi}$ is an interpretation when restricted to $\overleftrightarrow{\mathscr{A}\mathscr{B}}$, and the result follows from the following important lemma:

Lemma. Let $\mathscr{A}$ be a premanual, $\mathscr{B}$ a manual. If $\mathscr{A} \xrightarrow{\varphi} \mathscr{B}$ is an interpretation, then φ extended to $\langle \mathscr{A} \rangle$ is also an interpretation.

Proof. Let $X = \cup \mathscr{A}$ and define $\mathscr{S} = \{S \subseteq X$: if $S = A \dot{\cup} B$, then $\varphi(A) \text{oc}\, \varphi(B)$ in $\mathscr{B}\}$. Since φ is an interpretation on $\mathscr{A}$, we have $\mathscr{A} \subseteq \mathscr{S}$. *Claim*: $\mathscr{S}$ is a manual. Let $A \,\text{op}\, B$, $B \,\text{oc}\, C$ in $\mathscr{S}$, and let D be an axis for $A \,\text{op}\, B$. By the definition of $\mathscr{S}$, we have $\varphi(A) \text{op}\, \varphi(B)$, $\varphi(B) \text{oc}\, \varphi(C)$ in $\mathscr{B}$. Since $\mathscr{B}$ is a manual, $\varphi(A) \text{oc}\, \varphi(C)$ in $\mathscr{B}$. Now let $A \cup C = U \dot{\cup} V$. We must show that $\varphi(U) \text{oc}\, \varphi(V)$ in $\mathscr{B}$. Since $\varphi(U) \cup \varphi(V) = \varphi(A) \cup \varphi(C) \in \mathscr{B}$, it suffices to show that $\varphi(U) \cap \varphi(V) = \varnothing$. Suppose there is $x \in \varphi(U) \cap \varphi(V)$. Then there is $u \in U$, $\upsilon \in V$ with $x \in \varphi(u) \cap \varphi(\upsilon)$. Recall that $\varphi(A) \cap \varphi(C) = \varnothing$ so that either $u, \upsilon \in A$ or $u, \upsilon \in C$. Without loss of generality, say, $u, \upsilon \in A$. Now we partition $A \cup D$ into $W_1 \dot{\cup} W_2$ with $u \in W_1$, $\upsilon \in W_2$. Since $A \cup D \in \mathscr{S}$ by assumption, we have $\varphi(W_1) \text{oc}\, \varphi(W_2)$ which implies $x \perp x$, a contradiction. Hence, $\varphi(U) \text{oc}\, \varphi(V)$ in $\mathscr{B}$ and so $A \dot{\cup} C \in \mathscr{S}$ and the claim is proved. Thus $\langle \mathscr{A} \rangle \subseteq \mathscr{S}$, and it follows that $\langle \mathscr{A} \rangle \xrightarrow{\varphi} \mathscr{B}$ is an interpretation. ∎

We complete this section with several algebraic results involving the tensor product. We are first interested in the question of whether the tensor product "factors through" direct sums or products. This leads us to a generalization of the operational product, called a *Dacey sum*:

Definition 6.4. Let $\mathscr{I}$ be a manual with $I = \cup \mathscr{I}$. For every $i \in I$, let $\mathscr{A}_i$ be manual. Then the *Dacey sum of* $\mathscr{A}_i$ *over* $\mathscr{I}$ is equal to $\{ \cup_{e \in E} eF_e / E \in \mathscr{I}, F_e \in \mathscr{A}_e \}$.

It can be shown that the Dacey sum is always a manual. It is not hard to show that the following fact is true:

Proposition 6.5. Let $\{ \mathscr{A}_i \colon i \in I \}$ be a collection of manuals. In the category of manuals and interpretations, (i) the direct sum (or coproduct) of $\{ \mathscr{A}_i \colon i \in I \}$ is the Dacey sum of $\mathscr{A}_i$ over $\mathscr{I}$, where $\mathscr{I}$ is the transformation manual with outcome set I. The sum is denoted ΣA_i; (ii) the direct product of $\{ \mathscr{A}_i \colon i \in I \}$ is the Dacey sum of $\mathscr{A}_i$ over $\mathscr{I}$, where $\mathscr{I}$ is the classical manual with outcome set I. The product is denoted $\oplus \mathscr{A}_i$.

It is not hard to show that the tensor product does not factor through direct sums. That is, requiring only that $\mathscr{A}$ not be a transformation manual and the indexing set I have $\# I \geqslant 2$, it is always true that $\mathscr{A} \otimes (\Sigma \mathscr{B}_i) \neq \Sigma(\mathscr{A} \otimes \mathscr{B}_i)$. The question of whether or not the tensor product factors through the direct product is more difficult. The answer is yes if the indexing set is finite, but is no in general. Both the counterexample in the infinite case and the proof in the finite case require work beyond the scope of this paper, and are omitted here.

By definition, the tensor product of manuals is obviously commutative; that is, if $\mathscr{A} \otimes \mathscr{B}$ is defined, then $\mathscr{B} \otimes \mathscr{A}$ is defined and $\mathscr{A} \otimes \mathscr{B} \simeq \mathscr{B} \otimes \mathscr{A}$. Determining whether or not the tensor product is associative is quite difficult, and is still an open question. However, the answer is known in the locally finite case:

> *Theorem 6.6.* Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be locally finite manuals. Then if one of $\mathscr{A} \otimes (\mathscr{B} \otimes \mathscr{C})$ or $(\mathscr{A} \otimes \mathscr{B}) \otimes \mathscr{C}$ is defined, then the other one is, and $\mathscr{A} \otimes (\mathscr{B} \otimes \mathscr{C}) = (\mathscr{A} \otimes \mathscr{B}) \otimes \mathscr{C}$.

The proof of this theorem is based on two lemmas, each proved in the locally finite case. The first states that for premanuals $\mathscr{A}$ and $\mathscr{B}$, $\langle \overleftrightarrow{\mathscr{A}\mathscr{B}} \rangle = \langle \overleftrightarrow{\langle \mathscr{A} \rangle \mathscr{B}} \rangle$. The second states that for manuals $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, $\langle \overleftrightarrow{(\overleftrightarrow{\mathscr{A}\mathscr{B}})\mathscr{C}} \rangle \simeq \langle \overleftrightarrow{\mathscr{A}(\overleftrightarrow{\mathscr{B}\mathscr{C}})} \rangle$. In each equality, the existence of one implies the existence of the other.

7. CONCLUDING REMARKS

It has been our intention to exhibit the basic ideas and definitions of a tensor product of generalized sample spaces and to discuss questions on the existence, the weight structure, and the categorical properties of this product. Among other topics in this area, we mention the following two. Much work has been done (and much remains to be done) on determining which operations are actually contained in the tensor product and obtaining a workable method for constructing them. Secondly, if we consider a manual modulo its equivalence relation, op, we obtain the *logic* of a manual. Much has been written about the so-called quantum logics and the question of how to tensor two logics (see Zecca, 1978). We believe that some insight into these problems can be obtained by studying the logics of manuals and their tensor products.

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